Tracking at the LHC

Seminar

Alexandros Attikis[1]

[1]University Of Cyprus

Wednesday 27th May, 2015

Supported by
FP7-PEOPLE-2013-IEF
grant "HLTaus" (625892)

The momentum of a charged particle can be calculated by measuring the bending in a magnetic field:

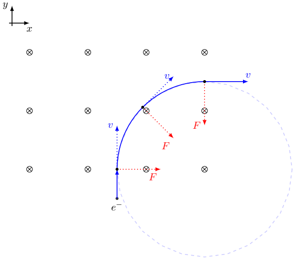
Lorentz Force Law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

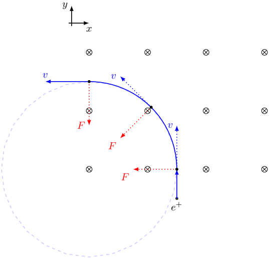
Usually $\vec{E} \approx 0$ inside detectors and thus has negligible effects on tracks, but $\vec{E} > 0$ is necessary for ionisation charge collection. The bending of the trajectory is thus solely due to $\vec{B}$ field.

Since $\vec{B}$ is at each point perpedingular to the charge's motion, the Lorentz force cannot do work on the moving charge. The B-field cannot speed up or slow down the charge, but only change its direction.

electron



positron



For a magnetic field $\vec{B} = (0, 0, B)$ and a track position vector $\vec{r} = (x, y, z)$ we have:

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = q\left(\vec{v} \times \vec{B}\right) = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \\ 0 & 0 & B \end{vmatrix}$$

$$m\left(\ddot{x}(t)\,\hat{i} + \ddot{y}(t)\,\hat{j} + \ddot{z}(t)\,\hat{k}\right) = qB\left(\dot{y}(t)\,\hat{i} - \dot{x}(t)\,\hat{j} + -0\hat{k}\right)$$

By equating components we get:

$$m\ddot{x}(t) = +qB\dot{y}(t)$$
$$m\ddot{y}(t) = -qB\dot{x}(t)$$
$$m\ddot{z}(t) = 0$$

The first two equations are coupled equations that define circular motion in the x-y plane. The third equation says that there is no acceleration parallel to the B-field. The equations of motion are:

$$\ddot{x}(t) = +\omega\dot{y}(t) \tag{1}$$
$$\ddot{y}(t) = -\omega\dot{x}(t) \tag{2}$$
$$\ddot{z}(t) = 0 \tag{3}$$

where $\omega = qBm^{-1}$ is the cyclotron frequency.

Integrating Eq. (1) with respect to time:

$$\int \ddot{x}(t)\,dt = +\omega \int \dot{y}(t)\,dt$$

$$\dot{x}(t) = +\omega y(t) + v_{x_0} \tag{4}$$

Integrating Eq. (2) with respect to time:

$$\int \ddot{y}(t)\,dt = -\omega \int \dot{x}(t)\,dt$$

$$\dot{y}(t) = -\omega x(t) + v_{y_0} \tag{5}$$

Integrating Eq. (3) twice with respect to time:

$$\int \ddot{z}(t)\,dt = \int 0 dt$$

$$\dot{z}(t) = v_{z_0}$$

$$\int \dot{z}(t)\,dt = \int v_{z_0}\,dt$$

$$z(t) = v_{z_0} t + z_0 \tag{6}$$

We can decouple Eq. (1) by substituting into it Eq. (5):

$$\ddot{x}(t) = +\omega \dot{y}(t)$$
$$\ddot{x}(t) = +\omega \left[-\omega x(t) + v_{y_0}\right]$$
$$\ddot{x}(t) = -\omega^2 x(t) + \omega v_{y_0}$$
$$\ddot{x}(t) + \omega^2 x(t) = \omega v_{y_0} \tag{7}$$

This is a nonhomogeneous 2nd order differential equation. The complementary equation $\ddot{x}(t) + \omega^2 x(t) = 0$ can be solved to obtain the complementary solution:

$$m^2 = -\omega^2$$
$$m = \sqrt{-\omega^2} = 0 \pm i\omega$$
$$x_c(t) = Ae^{0t}\cos(\omega t + \phi) = A\cos(\omega t + \phi) \tag{8}$$

Then, attempt to guess the particular solution as $x_p(t) = x_0$:

$$x(t) = x_c(t) + x_p(t)$$
$$x(t) = A\cos(\omega t + \phi) + x_0 \tag{9}$$
$$\Rightarrow \dot{x}(t) = -\omega A \sin(\omega t + \phi) \tag{10}$$
$$\Rightarrow \ddot{x}(t) = -\omega^2 A\cos(\omega t + \phi) \tag{11}$$

Check that Eq. (7) is satisfied by Eqs. (9)-(11):

$$\begin{aligned} \ddot{x}(t) + \omega^2 x(t) &= \omega v_{y_0} \\ -\omega^2 A\cos(\omega t + \phi) + \omega^2 \left[A\cos(\omega t + \phi) + x_0\right] &= \omega v_{y_0} \\ \omega^2 x_0 &= \omega v_{y_0} \\ x_0 &= \frac{v_{y_0}}{\omega} \\ \Rightarrow x(t) = A\cos(\omega t + \phi) + x_0 &= A\cos(\omega t + \phi) + \frac{v_{y_0}}{\omega} \qquad (12) \end{aligned}$$

Now, substitue Eq. (9) into Eq. (2):

$$\begin{aligned} \ddot{y}(t) &= -\omega \dot{x}(t) \\ \ddot{y}(t) &= -\omega \frac{d}{dt}\left[A\cos(\omega t + \phi) + x_0\right] \\ \ddot{y}(t) &= -\omega\left[-\omega A\sin(\omega t + \phi)\right] \\ \ddot{y}(t) &= +\omega^2 A\sin(\omega t + \phi) \\ \int \ddot{y}(t)\, dt &= +\omega^2 A \int \sin(\omega t + \phi)\, dt \\ \dot{y}(t) &= -\frac{\omega^{\not 2}}{\not\omega} A\cos(\omega t + \phi) + v_{y_0} \quad \ldots \end{aligned}$$

... From previous page:

$$\dot{y}(t) = -\omega A \cos(\omega t + \phi) + v_{y_0}$$

$$\int \dot{y}(t)\, dt = \int \left[-\omega A \cos(\omega t + \phi) + v_{y_0} \right] dt$$

$$y(t) = -\frac{\not{\omega} A}{\not{\omega}} \sin(\omega t + \phi) + v_{y_0} t + y_0$$

$$y(t) = -A \sin(\omega t + \phi) + v_{y_0} t + y_0 \quad (13)$$

$$\Rightarrow \dot{y}(t) = -\omega A \cos(\omega t + \phi) + v_{y_0} \quad (14)$$

$$\Rightarrow \ddot{y}(t) = +\omega^2 A \sin(\omega t + \phi) \quad (15)$$

Check that Eq. (1) is satisfied by Eq. (11) Eq. (12) and Eq. (13):

$$\ddot{x}(t) = +\omega \dot{y}(t)$$

$$-\omega^2 A \cos(\omega t + \phi) = +\omega \left[-\omega A \cos(\omega t + \phi) + v_{y_0} \right]$$

$$-\omega^2 A \cos(\omega t + \phi) = -\omega^2 A \cos(\omega t + \phi) + \omega v_{y_0}$$

$$\Rightarrow v_{y_0} = 0$$

$$y(t) = -A \sin(\omega t + \phi) + y_0 \quad (16)$$

We can now group together Eq. (6), Eq. (9), and Eq. (16) ...

The time-dependent positions of the charged particle in a constant B-field are given by:

Helix Equations

$$x(t) = +A\cos(\omega t + \phi) + x_0$$
$$y(t) = -A\sin(\omega t + \phi) + y_0$$
$$z(t) = v_{z_0} t + z_0$$
$$\omega = q\frac{B}{m} \qquad \text{(Cyclotron Frequency)}$$

The equations of motion in the x-y plane are:

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi)$$
$$\dot{y}(t) = -\omega A\cos(\omega t + \phi)$$
$$\dot{x}(t)^2 + \dot{y}(t)^2 = (-\omega A)^2\left[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right] = (\omega A)^2 = \left[\left(q\frac{B}{m}\right)A\right]^2 \tag{17}$$

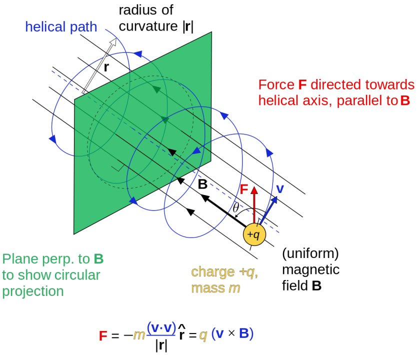
This is the equation for a circle of radius $\rho = \left(\frac{qB}{m}\right)A$. The motion in the z-axis is constant with time:

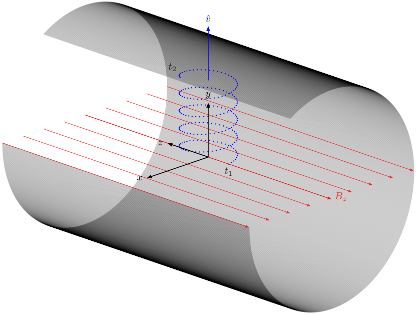
$$z(t) = v_{z_0} t + z_0$$
$$\Rightarrow \dot{z}(t) = v_{z_0} \tag{18}$$

Lorentz Force in Constant Magnetic Field

The charged particle traces a circle in the x-y plane, while also moving with constant speed along the z-axis. We therefore have a helix motion:

$$\dot{r}\,(t)^2 = \dot{x}\,(t)^2 + \dot{y}\,(t)^2 = \text{constant}$$
$$\dot{z}\,(t) = \text{constant}$$





A charged particle in a constant magnetic field experiences a centripetal acceleration due to the Lorentz force which causes it to move in helix motion:

$$F_c = F_L$$
$$\frac{mv^2}{\rho} = qvB$$

We can use the above expression to write the momentum of the particle as a function of the magnetic field and the radius of curvature of the track alone:

$$\frac{mv^2}{\rho} = qvB$$
$$\frac{(mv)\,v}{\rho} = qvB$$
$$\frac{p\not{v}}{\rho} = q\not{v}B$$
$$p[\text{kgms}^{-1}] = q[\text{C}] \cdot B[\text{T}] \cdot \rho[\text{m}]$$

where p is the particle momentum, B is the magnetic field strength and ρ is tha radius of curvature of the particle's trajectory.

It is convenient to convert the momentum in units of GeVc^{-1}:

$$
\begin{aligned}
c\left(p[\mathrm{kgms}^{-1}]\right) &= c\,(q[\mathrm{C}]\cdot B[\mathrm{T}]\cdot\rho[\mathrm{m}]) \\
pc[\mathrm{J}] &= ceB\rho && (E = pc) \\
pc[\mathrm{eV}] &= \frac{ceB\rho}{1.6\times10^{-19}} && (1\mathrm{eV} = 1.6\times10^{-19}\mathrm{J}) \\
pc[\mathrm{eV}] &= \frac{c\cancel{e}B\rho}{\cancel{1.6\times10^{-19}}} && (e = 1.6\times10^{-19}\mathrm{C}) \\
pc[\mathrm{eV}] &= 3\times10^{8}\cdot B\rho \\
pc[\mathrm{GeV}] &= \frac{3\times10^{8}}{1\times10^{9}}\cdot B\rho && (1\mathrm{GeV} = 1\times10^{9}\mathrm{eV}) \\
pc[\mathrm{GeV}] &= 0.3\cdot B[\mathrm{T}]\cdot\rho[\mathrm{m}] \\
\Rightarrow p[\mathrm{GeV}c^{-1}] &= 0.3\cdot B[\mathrm{T}]\cdot\rho[\mathrm{m}] && (19)
\end{aligned}
$$

Now, since the z-component of the momentum is zero then $|p| = |p_{\mathrm{T}}|$ and we can finally write:

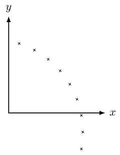
Transverse Momentum

$$p_{\mathrm{T}}[\mathrm{GeV}c^{-1}] = 0.3\cdot B[\mathrm{T}]\cdot\rho[\mathrm{m}]$$

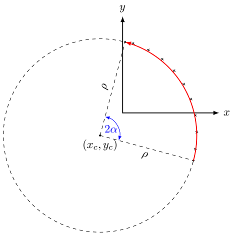
[Although derived using classical mechanics it is also valid for the HEP case, provided we use $p = \gamma m_0 v$.]

As the charged particle traverses the tracker layers, it interacts with it leaving behind detector "hits" which can be used to fit a circular trajectory to the track.

Tracker Hits

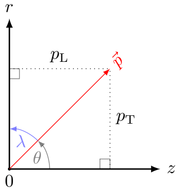


Track Fit

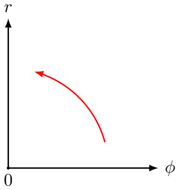


The momentum of the particle is projected along the longitudinal and transverse directions:

p_{L} in the ρ-z plane



p_{T} in the ρ-ϕ plane



From Eq. (19) we saw that the p_{T} of a track can be determined by measuring its radius of curvature (ρ). Tracking detectors measure positions of the track along various points along the track. By fitting a circle through all the measurement points the radius of curvature can be determined.

The sagitta (s) of a track is a measure of the bending of the trajectory and is used to determine ρ.

$$\sin\alpha = \frac{\frac{L}{2}}{\rho} = \frac{L}{2\rho}$$

$$\cos\alpha = \frac{\rho - s}{\rho}$$

$$\cos\alpha^2 + \sin\alpha^2 = 1$$

$$\left(\frac{\rho - s}{\rho}\right)^2 + \left(\frac{L}{2\rho}\right)^2 = 1$$

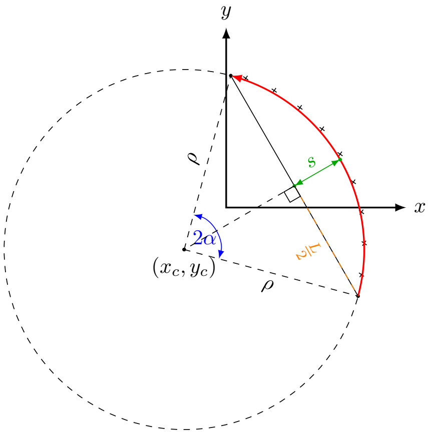
$$\frac{\rho^2 - 2\rho s + s^2}{\rho^2} + \frac{L^2}{2^2\rho^2} = 1$$

$$\cancel{\rho^2} - 2\rho s + s^2 + \frac{L^2}{4} = \cancel{\rho^2}$$

$$2\rho s = s^2 + \frac{L^2}{4}$$

$$\rho = \frac{L^2}{8s} + \frac{s}{2}$$

Track Sagitta



The radius of curvature (ρ) using the sagitta (s) and chord length (L) is given by:

Track Curvature From Sagitta

$$\rho = \frac{L^2}{8s} + \frac{s}{2}$$

$$\rho \approx \frac{L^2}{8s} \qquad (L \gg s)$$

We can differentiate the expression to see that when the curvature ρ increases, the saggita s decreases:

$$d\rho = -\frac{L^2}{8s^s} ds = -\frac{1}{s}\left(\frac{L^2}{8s^s}\right) ds \qquad (20)$$

$$d\rho = -\frac{\rho}{s} ds \qquad (d\rho \propto dp_{\mathrm{T}})$$

In other words, large momentum particles have large radius and small sagitta. The precision on the saggita measurement is the limiting factor of the momentum resolution.

From the sagitta equation we can of course derive an expression for the transverse momentum:

$$p_{\mathrm{T}}[\mathrm{GeV}c^{-1}] = 0.3 \cdot B[\mathrm{T}] \cdot \rho[\mathrm{m}]$$

$$p_{\mathrm{T}}[\mathrm{GeV}c^{-1}] = 0.3 \cdot B \cdot \left(\frac{L^2}{8s}\right)$$

The precision on the saggita measurement is the limiting factor of the momentum resolution:

$$p_{\mathrm{T}} = 0.3 \cdot B \cdot \left(\frac{L^2}{8s}\right)$$

$$\sigma^2_{p_{\mathrm{T}}} = \left(\frac{\partial p_{\mathrm{T}}}{\partial B}\right)^2 \sigma_B^2 + \left(\frac{\partial p_{\mathrm{T}}}{\partial L}\right)^2 \sigma_L^2 + \left(\frac{\partial p_{\mathrm{T}}}{\partial s}\right)^2 \sigma_s^2$$

$$\frac{\partial p_{\mathrm{T}}}{\partial B} = +\frac{p_{\mathrm{T}}}{B} \quad , \quad \frac{\partial p_{\mathrm{T}}}{\partial L} = +\frac{p_{\mathrm{T}}}{L} \quad , \quad \frac{\partial p_{\mathrm{T}}}{\partial s} = -\frac{p_{\mathrm{T}}}{s}$$

$$\Rightarrow \sigma^2_{p_{\mathrm{T}}} = \left(\frac{p_{\mathrm{T}}}{B}\right)^2 \sigma_B^2 + \left(\frac{p_{\mathrm{T}}}{L}\right)^2 \sigma_L^2 + \left(\frac{p_{\mathrm{T}}}{s}\right)^2 \sigma_s^2$$

$$\left(\frac{\sigma_{p_{\mathrm{T}}}}{p_{\mathrm{T}}}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2$$

It is convenient to write the momentum resolution only as a function of L and B:

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\frac{\sigma_s}{\frac{L^2}{8\rho}}\right)^2 = \left(\frac{\sigma_s}{L^2} 8\rho\right)^2 \qquad \left(s = \frac{L^2}{8\rho}\right)$$

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\sigma_s \frac{8}{L^2} \rho\right)^2 = \left(\sigma_s \frac{8}{L^2} \frac{p_{\mathrm{T}}}{0.3B}\right)^2 \qquad \left(\frac{p_{\mathrm{T}}}{0.3B} = \rho\right)$$

$$\Rightarrow \left(\frac{\sigma_s}{s}\right)^2 = \left(\sigma_s \frac{8 p_{\mathrm{T}}}{0.3 B L^2}\right)^2 \qquad (21)$$

Ignoring deviations from Multiple Coulomb Scattering (MCS), the momentum resolution is thus given by:

$$\left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2$$

$$\left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2 = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2 + \left(\sigma_s \frac{8p_\mathrm{T}}{0.3BL^2}\right)^2$$

Thus, sagitta resolution is closely linked to detector momentum resolution:

$$\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}^2} \propto \frac{\sigma_s}{BL^2}$$

The important features of the momentum resolution can be summarised as follows:

- the percentage error on the momentum is proportional to the momentum itself ($\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}} \propto p_\mathrm{T}$)
- the error on the momentum is inversely proportional to the magnetic field strength ($\sigma_{p_\mathrm{T}} \propto B^{-1}$)
- the error on the momentum is inversely proportional to the square of the chord length ($\sigma_{p_\mathrm{T}} \propto L^{-2}$)
- the error on the momentum is proportional to (sagitta) coordinate measurement error ($\sigma_{p_\mathrm{T}} \propto \sigma_s$)

The sagitta s is determined by measurements along the particle trajectory. Assume that we measure the y-coordinate at 3 equi-spaced measurements in (x, y) plane $(z = 0)$. Each y measurement has precision σ_y and we have a constant magnetic field $\vec{B} = (0, 0, B)$. The sagitta and its error are given by:

$$s = y_2 - \frac{y_1 + y_3}{2}$$

$$\sigma_s^2 = \sum_{i=1}^{i=3} \left(\frac{\partial s}{\partial y_i}\right)^2 \sigma_{y_i}^2$$

$$\sigma_s^2 = \left(\frac{\partial s}{\partial y_1}\right)^2 \sigma_{y_1}^2 + \left(\frac{\partial s}{\partial y_2}\right)^2 \sigma_{y_2}^2 + \left(\frac{\partial s}{\partial y_3}\right)^2 \sigma_{y_3}^2$$

$$\frac{\partial s}{\partial y_1} = -\frac{1}{2} \quad ; \quad \frac{\partial s}{\partial y_2} = 1 \quad ; \quad \frac{\partial s}{\partial y_3} = -\frac{1}{2}$$

$$\sigma_s^2 = \left(-\frac{1}{2}\right)^2 \sigma_{y_1}^2 + (1)^2 \sigma_{y_2}^2 + \left(-\frac{1}{2}\right)^2 \sigma_{y_3}^2$$

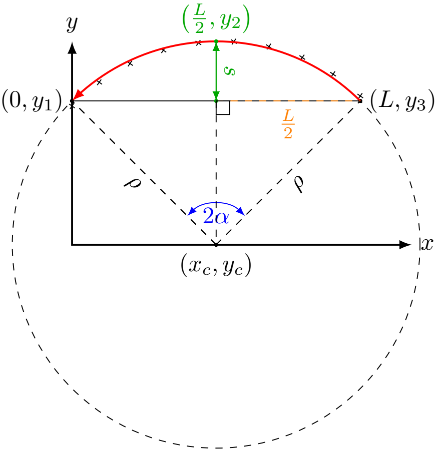
$$\sigma_s^2 = \frac{1}{4}\sigma_{y_1}^2 + \sigma_{y_2}^2 + \frac{1}{4}\sigma_{y_3}^2$$

$$\sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} = \sigma_y$$

$$\sigma_s^2 = \frac{6}{4}\sigma_y^2 = \frac{3}{2}\sigma_y^2$$

$$\Rightarrow \sigma_s = \sqrt{\frac{3}{2}}\sigma_y \qquad (22)$$

Sagitta Measurement



From Eq. (21) we saw that:

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\sigma_s \frac{8p_{\mathrm{T}}}{0.3BL^2}\right)^2$$

Substitute Eq. (22) in the above equation to express the sagitta error in term of the position resolution:

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\sigma_s \frac{8p_{\mathrm{T}}}{0.3BL^2}\right)^2 = \left(\sqrt{\frac{3}{2}}\sigma_y \cdot \frac{8p_{\mathrm{T}}}{0.3BL^2}\right)^2$$

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\sqrt{\frac{192}{2}}\frac{\sigma_y p_{\mathrm{T}}}{0.3BL^2}\right)^2 \qquad \text{(3 measurements)}$$

$$\left(\frac{\sigma_s}{s}\right)^2 = \left(\sqrt{\frac{720}{N+4}}\frac{\sigma_y p_{\mathrm{T}}}{0.3BL^2}\right)^2 \qquad \text{(N measurements)}$$

Momentum Resolution (Measurement)

$$\left(\frac{\sigma_{p_{\mathrm{T}}}}{p_{\mathrm{T}}}\right)^2_{\text{meas.}} = \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2 + \left(\sqrt{\frac{720}{N+4}}\frac{\sigma_y p_{\mathrm{T}}}{0.3BL^2}\right)^2 \qquad \text{(N measurements)}$$

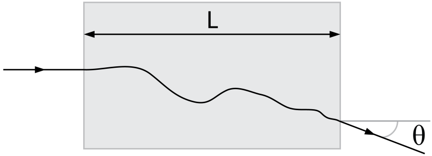
where σ_y is the spatial resolution of our tracker modules.

A particle with charge number z traversing a medium is deflected by many small-angle scatters, though elastic scattering with a nucleus of atomic number Z. The differential cross-section for this EM processes known as Rutherford (Coulomb) scattering is:

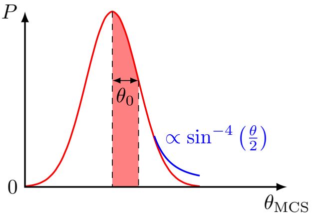
$$\frac{d\sigma}{d\Omega} = 4zZr_e^2 \left(\frac{m_e c}{\beta p} \right) \frac{1}{\sin^4 \frac{\theta}{2}} \tag{23}$$

which is non-relativistic and does not account for spin. After passing material layer of thickness L it leaves with some displacement from the incident direction and some deflection angle θ. For hadronic projectiles, strong interactions also contribute to MCS.

MCS



MCS Deflection Angle



The distribution of the deflection angle is roughly Gaussian for small deflection angles but at larger angles it behaves like Rutherford scattering (larger tails).

Tracking algorithms assume that the MCS angle follows a Gaussian distribution whose RMS is given by:

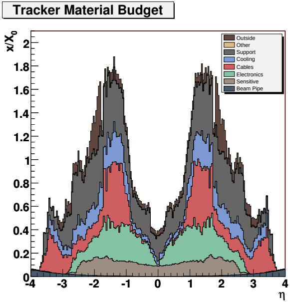
Highland Formula

$$\theta_0 = \frac{13.6\text{MeV}}{\beta cp} z \sqrt{\frac{x}{X_0}} \left[1 + 0.038 \ln \left(\frac{x}{X_0} \right) \right]$$

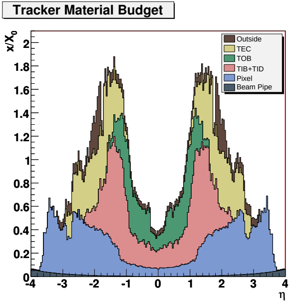
where X_0 is radiation length of the scattering medium; the mean distance over which the energy of a high-energy electron drops to $1/e$ by bremsstrahlung since $E(x) = e^{-\frac{x}{X_0}}$.

It is thus evident that the amount of tracker material affects the track reconstruction:

CMS Phase-0 Tracker



CMS Phase-0 Tracker



The MCS deflects the tracks and it affects the detector residuals which become momentum dependent.

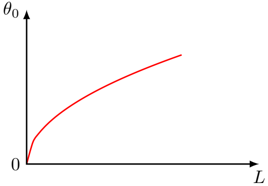
In a material of radiation length X_0, a charged particle with momentum p will undergo MCS. After passing layer of thickness L it emerges laterally displaced by a distance ε and deflected by an angle θ:

MCS Deflection Angle

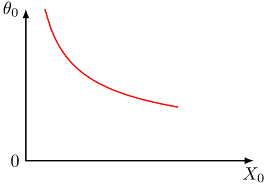
$$\theta_0 \propto \frac{1}{p}\sqrt{\frac{L}{X_0}}$$

θ_0 Vs. Momentum

θ_0 Vs. Material Thickness



θ_0 Vs. Radiation Length



The important features of the MCS deflection angle can be summarised as follows:

- Proportional to the material thickness traversed ($\theta_0 \propto \sqrt{L}$)
- Inversely proportional to the momentum ($\theta_0 \propto p^{-1}$); the effect is 10× larger for a 2 GeV pion than for a 20 GeV pion.
- Inversely proportional to the material's radiation length ($\theta_0 \propto \frac{1}{\sqrt{X_0}}$);

MCS dominates the momentum measurement resolution for low momenta.

At low p_T the momentum resolution is dominated by MCS whose contribution is:

$$\theta_0 = \frac{13.6\mathrm{MeV}}{\beta cp} z \sqrt{\frac{x}{\mathrm{X}_0}} \left[1 + 0.038 \ln\left(\frac{x}{\mathrm{X}_0}\right)\right]$$

$$\sigma_\theta \approx \frac{13.6\mathrm{MeV}}{p} \sqrt{\frac{L}{\mathrm{X}_0}} \qquad (x = L)$$

$$\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}} = \frac{\sigma_\rho}{\rho} = \frac{\sigma_\theta}{\theta} \qquad (24)$$

$$\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}} = \frac{\sigma_\theta}{\theta} = \frac{13.6\mathrm{MeV}}{p} \sqrt{\frac{L}{\mathrm{X}_0}} \frac{1}{\theta} = \frac{13.6\mathrm{MeV}}{p} \sqrt{\frac{L}{\mathrm{X}_0}} \frac{\rho}{L} \qquad (L \simeq \rho\theta)$$

$$\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}} = \frac{13.6\mathrm{MeV}}{\cancel{p_\mathrm{T}}} \sqrt{\frac{L}{\mathrm{X}_0}} \frac{\cancel{p_\mathrm{T}}}{0.3BL} = 13.6\mathrm{MeV} \sqrt{\frac{\cancel{L}}{\mathrm{X}_0 L^{\cancel{2}}}} \frac{1}{0.3B} \qquad (p_\mathrm{T} = 0.3B\rho)$$

$$\left.\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right|_{\mathrm{MCS}} = \frac{13.6\mathrm{MeV}}{0.3} \frac{1}{B\sqrt{\mathrm{X}_0 L}}$$

Momentum Resolution (Multiple Coulomb Scattering)

$$\left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2_{\mathrm{MCS}} \propto \left(\frac{1}{B\sqrt{L\mathrm{X}_0}}\right)^2 \qquad (\text{constant})$$

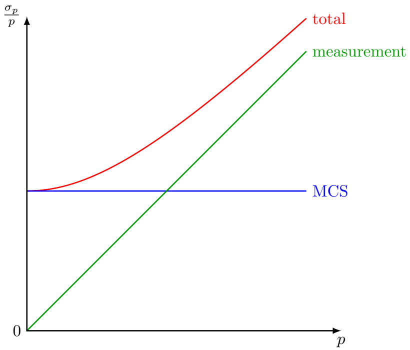
which means that the MCS contribution to the p_T resolution is independent of the momentum!

Momentum Resolution (Total)

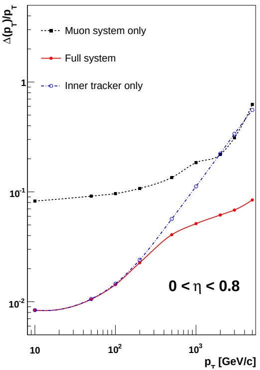
$$\left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2_{\text{total}} = \left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2_{\text{meas.}} + \left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2_{\text{MCS}} = \left(\sqrt{\frac{720}{N+4}}\frac{\sigma_y p_\mathrm{T}}{0.3BL^2}\right)^2 + \text{const}\cdot\left(\frac{1}{B\sqrt{L\mathrm{X}_0}}\right)^2$$

$$\left(\frac{\sigma_{p_\mathrm{T}}}{p_\mathrm{T}}\right)^2_{\text{total}} = \text{const}\cdot\left(\frac{p_\mathrm{T}}{BL^2}\right)^2 + \text{const}\cdot\left(\frac{1}{B\sqrt{L\mathrm{X}_0}}\right)^2$$

p_T resolution



CMS Phase-0



Momentum Measurement Limitation

Assume a cylindrical tracking system with a $R = 1$m radius barrel embedded in a uniform constant magnetic field of $B = 1$T along its long axis. What is the maximum transverse momentum that can be measured if the spatial resolution of the tracking system is 20 μm?

Momentum Measurement Limitation

Assume a cylindrical tracking system with a $R = 1$m radius barrel embedded in a uniform constant magnetic field of $B = 1$T along its long axis. What is the maximum transverse momentum that can be measured if the spatial resolution of the tracking system is 20 μm?

The transverse momentum of a charged particle in a uniform magnetic field is given by:

$$p_{\mathrm{T}} = 0.3 \cdot B[\mathrm{T}] \cdot \rho[\mathrm{m}] \qquad (\mathrm{GeV}c^{-1})$$

while the radius of curvature is given by:

$$\rho = \frac{L^2}{8s} + \frac{s}{2} \approx \frac{L^2}{8s} \qquad (L \gg s)$$

Combining the two results together we get:

$$p_{\mathrm{T}} = 0.3 \cdot B[\mathrm{T}] \cdot \left(\frac{L^2}{8s} + \frac{s}{2}\right) \approx 0.3 \cdot B[\mathrm{T}] \cdot \frac{L^2}{8s} \qquad (L \gg s)$$

- L is the size of the tracking system
- ρ and s is the radius of curvature and the sagitta of the track, respectively

Momentum Measurement Limitation

Assume a cylindrical tracking system with a $R = 1$m radius barrel embedded in a uniform constant magnetic field of $B = 1$T along its long axis. What is the maximum transverse momentum that can be measured if the spatial resolution of the tracking system is 20 μm?

The maximum possible track chord length L is the radius of the barrel ($L = L_{\max} = R$), and the smallest measurable sagitta is the spatial resolution ($s = s_{\min} = \sigma_s$).

This means that the maximum measurable momentum is:

$$p_{\mathrm{T,\ max}} \approx 0.3 \cdot B[\mathrm{T}] \cdot \frac{L^2_{\max}}{8 s_{\min}}[\mathrm{m}] \qquad (\mathrm{GeV}c^{-1})$$

$$p_{\mathrm{T,\ max}} \approx 0.3 \cdot B[\mathrm{T}] \cdot \frac{R^2}{8\sigma_s}[\mathrm{m}]$$

$$p_{\mathrm{T,\ max}} \approx 0.3 \cdot 1[\mathrm{T}] \cdot \frac{1^2}{8 \times 20 \times 10^{-6}}[\mathrm{m}]$$

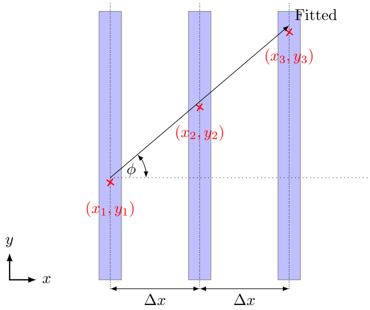
$$p_{\mathrm{T,\ max}} \approx \frac{0.3}{160} \times 10^6 = 1875 \qquad (\mathrm{GeV}c^{-1})$$

$$\Rightarrow p_{\mathrm{T,\ max}} \approx 1.88 \qquad (\mathrm{TeV}c^{-1})$$

Consider a simple tracking system built using 3 equidistant and identical planes. The incident charged particle traverses the layers in a straight path, leaving behind "layer hits" which correspond to measurements.

Using the 3 measurements we can perform a fit by computing the residuals that minimise the track χ^2.

Simple Tracking System



The track can be parametrised as:

$$y = y_0 + x \tan \phi$$

and thus the track parameters set is given by $\mathbf{t} = (\mathbf{y}_0, \tan \phi)$.

The residuals of the track in each plane can be derived from the χ^2 minimisation condition:

$$\chi^2 = \mathbf{r}^{\mathbf{T}} \cdot \mathbf{V}^{-\mathbf{1}} \cdot \mathbf{r} \tag{25}$$

$$\frac{d\chi^2}{d\mathbf{t}} = \left(\frac{d\mathbf{r}}{d\mathbf{t}}\right)^T \cdot V^{-1} \cdot r = 0. \tag{26}$$

If the vector of measurements is $\mathbf{m}$ (with m_i the measurement in plane i and the extrapolated points vector is $\mathbf{e}$, then vector of residuals ($\mathbf{r}$) is given by:

$$\mathbf{r} = \mathbf{m} - \mathbf{e}$$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} m_1 - (y_0 + x_1 \tan\phi) \\ m_2 - (y_0 + x_2 \tan\phi) \\ m_3 - (y_0 + x_3 \tan\phi) \end{pmatrix}$$

From Eq. (26) it can be seen that for the track fit the derivatives of the residuals are required:

$$\frac{d\mathbf{r}}{d\mathbf{t}} = \begin{pmatrix} \frac{dr_1}{dt_1} & \frac{dr_1}{dt_2} \\ \frac{dr_2}{dt_1} & \frac{dr_2}{dt_2} \\ \frac{dr_3}{dt_1} & \frac{dr_3}{dt_2} \end{pmatrix} = \begin{pmatrix} \frac{dr_1}{dy_0} & \frac{dr_1}{d\tan\phi} \\ \frac{dr_2}{dy_0} & \frac{dr_2}{d\tan\phi} \\ \frac{dr_3}{dy_0} & \frac{dr_3}{d\tan\phi} \end{pmatrix} = \begin{pmatrix} -1 & -x_1 \\ -1 & -x_2 \\ -1 & -x_3 \end{pmatrix}$$

From Eq. (26) we also see that the inverse covariance matrix V^{-1} is also needed. First thing to note is that if all the planes at x_1, x_2, and x_3 are identical they must have the same resolution:

$$V = \begin{pmatrix} COV(r_1, r_1) & COV(r_1, r_2) & COV(r_1, r_3) \\ COV(r_2, r_1) & COV(r_2, r_2) & COV(r_2, r_3) \\ COV(r_3, r_1) & COV(r_3, r_2) & COV(r_3, r_3) \end{pmatrix}$$

$$COV(r_\mathrm{i}, r_\mathrm{j}) = 0 \qquad \text{(for } i \neq j\text{)}$$

$$COV(r_\mathrm{i}, r_\mathrm{j}) = \sigma_i^2 = \sigma^2 \qquad \text{(for } i = j\text{)}$$

The error in plane i is the same for all values that i can take, since they are made of the same module technology. Therefore, the covariance matrix is diagonal:

$$V = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sigma^2 \mathbb{I}_3$$

In order to get the covariant matrix inverse we need to:

1. Ensure that $det(V) \neq 0$
2. Take transpose matrix V^T
3. Get cofactor matrix C
4. Get adjoint matrix $adj(V)$

1. Ensure that $det\left(V\right) \neq 0$:

$$det\left(V\right) = \sigma^2\left(\sigma^2 \cdot \sigma^2 - 0 \cdot 0\right) - 0\left(0 \cdot \sigma^2 - 0 \cdot 0\right) + 0\left(0 \cdot 0 - \sigma^2\right) = \sigma^6$$
$$\Rightarrow det\left(V\right) = \sigma^6$$

2. Take transpose matrix V^{T}:

$$V^{\mathrm{T}} = \sigma^2 {\mathbb{I}_3}^{\mathrm{T}} = \sigma^2 \mathbb{I}_3 = V$$
$$\Rightarrow V^{\mathrm{T}} = V$$

3. Get cofactor matrix C:

$$C = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} \left(\sigma^2 \cdot \sigma^2 - 0 \cdot 0\right) & \left(0 \cdot \sigma^2 - 0 \cdot 0\right) & \left(0 \cdot -\sigma^2 \cdot 0\right) \\ \left(0 \cdot \sigma^2 - 0 \cdot 0\right) & \left(\sigma^2 \cdot \sigma^2 - 0 \cdot 0\right) & \left(\sigma^2 \cdot 0 - 0 \cdot 0\right) \\ \left(0 \cdot 0 - 0 \cdot \sigma^2\right) & \left(\sigma^2 \cdot 0 - 0 \cdot 0\right) & \left(\sigma^2 \cdot \sigma^2 - 0 \cdot 0\right) \end{pmatrix}$$
$$\Rightarrow C = \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & \sigma^4 & 0 \\ 0 & 0 & \sigma^4 \end{pmatrix}$$

4. Get adjoint matrix $adj\,(V)$:

$$adj\,(V) = C \cdot \begin{pmatrix} + & - & + \\ + & - & + \\ + & - & + \end{pmatrix} = \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & \sigma^4 & 0 \\ 0 & 0 & \sigma^4 \end{pmatrix} \cdot \begin{pmatrix} + & - & + \\ + & - & + \\ + & - & + \end{pmatrix}$$

$$adj\,(V) = \begin{pmatrix} \sigma^4 & 0 & 0 \\ 0 & \sigma^4 & 0 \\ 0 & 0 & \sigma^4 \end{pmatrix} = \sigma^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow adj\,(V) = \sigma^4 \mathbb{I}_3$$

Then we finally have the inverse Covariant matrix:

$$V^{-1} = \frac{1}{det\,(V)} \cdot adj\,(V)$$

$$V^{-1} = \frac{1}{\sigma^6} \cdot \sigma^4 \mathbb{I}_3$$

$$\Rightarrow V^{-1} = \frac{1}{\sigma^2} \mathbb{I}_3$$

which we can use to perform the linearisation of the hits.

We can now apply Eq. (26):

$$\frac{d\chi^2}{d\mathbf{t}} = \left(\frac{d\mathbf{r}}{d\mathbf{t}}\right)^T \cdot V^{-1} \cdot r = 0$$

$$\frac{d\chi^2}{d\mathbf{t}} = \begin{pmatrix} -1 & -x_1 \\ -1 & -x_2 \\ -1 & -x_3 \end{pmatrix}^T \cdot \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \cdot \begin{pmatrix} m_1 - (y_0 + x_1 \tan\phi) \\ m_2 - (y_0 + x_2 \tan\phi) \\ m_3 - (y_0 + x_3 \tan\phi) \end{pmatrix} = 0$$

$$\frac{d\chi^2}{d\mathbf{t}} = \begin{pmatrix} -1 & -1 & -1 \\ -x_1 & -x_2 & -x_3 \end{pmatrix} \cdot \sigma^2 \mathbb{I}_3 \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 0$$

$$\Rightarrow 0 = \begin{pmatrix} +1 & +1 & +1 \\ +x_1 & +x_2 & +x_3 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

The problem can be simplified by setting the first plane at the origin so that $x_1 = 0$. Then, it follows that the planes coordinates are:

$$(x_1, y_1) = (0, y_1) \tag{27}$$

$$(x_2, y_2) = (\Delta x, y_2) \tag{28}$$

$$(x_3, y_3) = (2\Delta x, y_3) \tag{29}$$

We therefore have an equation system with 2 equations and 3 unknowns:

$$\begin{pmatrix} +1 & +1 & +1 \\ +x_1 & +x_2 & +x_3 \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Substitute the plane coordinates from Eq. (27):

$$\begin{pmatrix} +1 & +1 & +1 \\ 0 & \Delta x & 2\Delta x \end{pmatrix} \cdot \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} r_1 + r_2 + r_3 &= 0 \\ \Delta x \left(r_2 + 2r_3 \right) &= 0 \\ \Rightarrow r_2 &= -2r_3 \\ r_1 - r_3 &= 0 \\ \Rightarrow r_1 &= r_3 \end{aligned}$$

We thus have the relations connecting the residuals:

$$r_1 = r_3 \tag{30}$$

$$r_2 = -2r_3 = -2r_1 \tag{31}$$

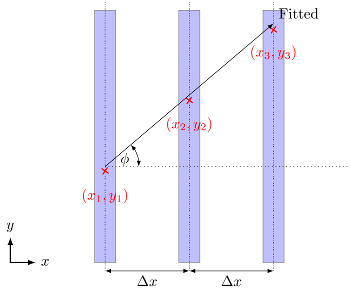
χ^2 Minimisation

$$\chi^2 = \mathbf{r}^\mathbf{T} \cdot \mathbf{V}^{-1} \cdot \mathbf{r} \qquad (\chi^2 \text{ Definition})$$

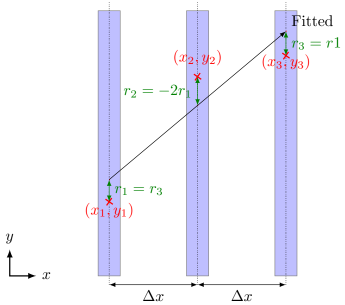
$$\mathbf{r} = \mathbf{m} - \mathbf{e} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} m_1 - e_1 \\ m_2 - e_2 \\ m_3 - e_3 \end{pmatrix} \qquad (\text{Residual Definition})$$

$$\frac{d\chi^2}{d\mathbf{t}} = \left(\frac{d\mathbf{r}}{d\mathbf{t}}\right)^T \cdot V^{-1} \cdot r = 0 \qquad (\text{Minimisation Condition})$$

Tracking Hits



Fit Residuals



The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° 625892.